3

Stock and Bond Valuation: Annuities and Perpetuities

Important Shortcut Formulas

The present value formula is the main workhorse for valuing investments of all types, including stocks and bonds. But these rarely have just two or three future payments. Stocks may pay dividends forever. The most common mortgage bond has 360 monthly payments. It would be possible but tedious to work with NPV formulas containing 360 terms.

Fortunately, there are some shortcut formulas that can speed up your PV computations if your projects have a particular set of cash flow patterns and the opportunity cost of capital is constant. The two most prominent are for projects called *perpetuities* (which have payments lasting forever) and *annuities* (which have payments lasting for a limited number of years). Of course, no firm lasts forever, but the perpetuity formula is often a useful "quick-and-dirty" tool for a good approximation. In any case, the formulas in this chapter are widely used and can help you understand the economics of corporate growth.

3.1 Perpetuities and Growing Perpetuities

A simple perpetuity is a project with a stream of constant cash flows that repeats forever. If the cost of capital (i.e., the appropriate discount rate) is constant and the amount of money remains the same or grows at a constant rate, perpetuities lend themselves to fast present value solutions — very useful when you need to come up with quick rule-of-thumb estimates. Though the formulas may seem intimidating at first, using them will quickly become second nature to you.

The Simple Perpetuity Formula

At a constant interest rate of 10%, how much money do you need to invest today to receive the same dollar amount of interest of $2 each year, starting next year, forever? Exhibit **??** shows the present values of all future payments for a perpetuity paying $2 forever, if the interest rate is 10% per annum. Note how there is no payment at time 0, and that the individual payment terms become smaller and smaller the further out we go.

(Omitted fig)

To confirm the table's last row, which gives the perpetuity's net present value as $20, you can spend from here to eternity to add up the infinite number of terms. But if you use a spreadsheet to compute and add up the first 50 terms, you will get a PV of $19.83. If you add up the first 100 terms, you will get a PV of $19.9986. Mathematically, the sum eventually converges to $20 sharp. This is because there is a nice shortcut to computing the net present value of the perpetuity if the cost of capital is constant:

(Omitted eq)

The "1" time subscript in the formula is to remind you that the first cash flow occurs not now, but next year — the cash flows themselves will remain the same amount next year, the year after, and so on.

Begin Important

A stream of constant cash flows (C dollars each period and forever) beginning *next* period (i.e., time 1), which is discounted at the same per-period cost of capital r forever, is a special perpetuity worth

(Omitted eq)

which is a shortcut for

(Omitted eq)

C_2 and all other C_t are the same as C_1. **End Important**

(Omitted anecdote)

The easiest way for you to get comfortable with perpetuities is to solve some problems.

(Omitted solvenow)

The Growing Perpetuity Formula

What if, instead of the same amount of cash every period, the cash flows increase over time? The growing perpetuity formula allows for a constant rate g per period, provided it is less than the interest rate. Exhibit **??** shows a growing perpetuity that pays $2 next year, grows at a rate of 5%, and faces a cost of capital of 10%. The present value of the first 30 terms adds up to $30.09. The first 100 terms add up to $39.64. The first 200 terms add up to $39.98. Eventually, the sum approaches the formula

(Omitted eq)

(Omitted fig)

As before, the "1" subscript indicates that cash flows begin next period, not this period, but here it is necessary because future cash flows will be different. The interest rate is r and it is reduced by g, the growth rate of your cash flows. Note how the table shows that the first application of the growth factor g occurs 1 period after the first application of the discount factor. For example, the cash flow at time 30 is discounted by $(1+r)^{30}$, but its cash flow is C multiplied by a growth factor of $(1+g)^{29}$. You will later encounter many applications of the growing perpetuity formula. For example, it is common to assume that cash flows grow by the rate of inflation.2]sect:terminalvalueTerminal value You will also later use this formula to obtain so-called terminal values in a chapter of this book, in which you design so-called pro formas.

Begin Important

A stream of cash flows growing at a rate of g each period and discounted at a constant interest rate r is worth

(Omitted eq)

The first cash flow, C_1, occurs next period (time 1), the second cash flow of $C_2 = C_1 \cdot (1+g)$ occurs in two periods, and so forth, *forever*. For the formula to work, g can be negative, but r must be greater than g.

You need to memorize the growing perpetuity formula! **End Important**

Be careful to use the cash flow *next* year in the numerator. The subscript "1" is there to remind you. For example, if you want to use this formula on your firm, and it earned $100 million this year, and you expect it to grow at a 5% rate forever, then the correct cash flow in the numerator is $C_1 = \$105$ million, not $100 million!

What would happen if the cash flows grew faster than the interest rate ($g > r$)? Wouldn't the formula indicate a negative PV? Yes, but this is because the entire scenario would be nonsense. The present value in the perpetuities formulas is only less than infinity, because *in today's dollars*, each term in the sum is a little less than the term in the previous period. If g were greater than r, however, the cash flow 1 period later would be worth more even in today's dollars. For example, take our earlier example with a discount rate of 10%, but make the growth rate of cash flows $g = 15\%$. The first cash flow would still be $2, which still discounts to $1.818 today. But the second cash flow would be $\$2 \cdot 1.15 = \$2.30$, which discounts to $1.901 today. The third cash flow would be $\$2 \cdot 1.15^2 = \$2.645$, which discounts to $1.987 today. The present value of each cash flow is higher than that preceding it. Taking a sum over an infinite number of such increasing terms would yield infinity as the value. A value of infinity is clearly not sensible, as nothing in this world is worth an infinite amount of money. Therefore, the growing perpetuity formula yields nonsensical values if $g \geq r$ — as it should!

(Omitted solvenow)

Application: Stock Valuation with A Gordon Growth Model

With their fixed interest and growth rates and eternal payment requirements, perpetuities are rarely exactly correct. But they can be very helpful for quick back-of-the-envelope estimates. For example, consider a mature and stable business with profits of $1 million next year. Because it is stable, its profits are likely to grow at the inflation rate of, say, 2% per annum. This means that it will earn $1,020,000 in 2 years, $1,040,400 in 3 years, and so on. The firm faces a cost of capital of 8%. The growing perpetuity formula indicates that this firm should probably be worth no more than

(Omitted eq)

because in reality, the firm will almost surely not exist forever. Of course, in real life, there are often even more significant uncertainties: Next year's profit may be different, the firm may grow at a different rate (or may grow at a different rate for a while) or face a different cost of capital for 1-year loans than it does for 30-year loans. Thus, $16.7 million should be considered a quick-and-dirty useful approximation, perhaps for an upper limit, and not an exact number.

The growing perpetuity model is sometimes directly applied to the stock market. For example, if you believe that a stock's dividends will grow by $g = 5\%$ forever, that the appropriate rate of return is $r = 10\%$, and that the stock will earn and/or pay dividends of $D = \$10$ *next year*, then you would feel that a stock price today of

(Omitted eq)

would be appropriate. In this context, the growing perpetuity model is often called the Gordon growth model, after its inventor, Myron Gordon.

Let us explore the Gordon growth model a bit. In late 2021, *YAHOO!*FINANCE stated that Walmart (WMT) had a forward-looking dividend yield of 2.2%. This is the analysts' consensus forecast of next year's dividends divided by the stock price, D/P. This is called the dividend yield. Rearrange Formula **??**:

(Omitted eq)

Therefore, you can infer that the market believes that the appropriate cost of capital (r) for Walmart's equity exceeds its growth rate of dividends (g) by about 2.2%. *YAHOO!*FINANCE further had a summary of Walmart's cash flow statement, which indicated that it paid $6.116 billion in cash dividends in 2021, almost the same as the $6.124 billion it paid in 2018. Therefore, if you believe 0%/year is also a fair estimate of the *eternal* future growth rate of Walmart's dividends, then the financial markets valued Walmart as if it had a per-annum cost of capital r of about

(Omitted eq)

Don't take this estimate too seriously. It is an approximation that should be viewed just as a conversation starter. (Neither the growth rate nor the cost of capital will remain constant forever.)

Let's play another game that is prominent in the financial world. Earnings are, loosely speaking, cousins of the cash flows that corporate stockholders receive. You can then think of the value of the stock today as the value of the earnings stream the stock will produce. After all, recall from Chapter that owners receive all dividends and all cash flows (earnings), presumably the former being paid out from the latter. (In Chapter , I will explain a lot of this in more detail as well as why earnings are only approximately but not exactly cash flows.)

Furthermore, it is common to assume that stock-market values are "capitalized" as if corporate earnings were eternal cash flows that are growing at a constant rate g applicable to earnings (which is not necessarily the same as the growth rate applicable to dividends). This means that you would assume that the value of the firm is

(Omitted eq)

Thus, to determine the rate of return that investors require (the cost of capital), all you need is a forecast of earnings, the current stock price, and the eternal growth rate of earnings. Again, YAHOO! FINANCE Analysis gives you all the information you need. In late 2021, Walmart's "trailing P/E" ratio — calculated as the current stock price divided by historical earnings — was 42. More interestingly, the analysts predicted "forward P/E" ratios — calculated as the price divided by their expectations of *next* year's earnings — was 22. The growing perpetuity formula wants the earnings in *future* years, so the latter is closer to what you need. The analysts also expected WMT's earnings to grow over the next 5 years at an average rate of 15% — the g in the formula if you are willing to assume that this is a long-term quasi-eternal growth rate. Therefore, all you have to do is rearrange the growing perpetuity formula, and out pops an appropriate rate of return for WMT's equity:

(Omitted eq)

As a herd, analysts were quite optimistic on WMT's earnings relative to its price and more so than they were with respect to how much it would pay out in dividends.

This formula is intuitive, but there are also more complex versions. For example, analysts sometimes use one that contemplates that firms with higher earnings reinvestment rates (aka plowback ratios) should have higher earnings growth rates g. (This seems to be the case here, too.)

It is important that you recognize these are just approximations that you should not take too seriously in terms of accuracy. Walmart will not last forever, earnings are not the cash flows you need, the discount rate is not eternally constant, earnings will not grow forever at 6.3%, and so on. However, the numbers are not uninteresting and may not even be too far off, either. Walmart is a very stable company that is likely to be around for a long time, but I would still not trust either of these cost-of-capital estimates. (Knowing more about the financial markets, one seems too low to me, the other too high.)

(Omitted solvenow)

3.2 Annuities

The second type of cash flow stream that lends itself to a quick formula is an annuity, which is a stream of equal cash flows for a given number of periods. Unlike a perpetuity, payments stop after T periods. For example, if the interest rate is 10% per period, what is the value of an annuity that pays $5 per period for 3 periods?

Let's first do this the slow way. You can hand-compute the net present value as

(Omitted eq)

The annuity formula makes short work of this NPV calculation,

(Omitted eq)

Is this really a shortcut? Maybe not for 3 periods, but try a 360-period annuity — which method do you prefer? Either works.

Begin Important

A stream of constant equal cash flows, beginning next period (time 1) and lasting for T periods, and discounted at a constant interest rate r, is worth

(Omitted eq)

You need to memorize the annuity formula! **End Important**

(Omitted solvenow)

(Omitted anecdote)

Annuity Application: Fixed-Rate Mortgage Payments

Most mortgages are fixed-rate mortgage loan[fixed-rate mortgage loans], and they are basically annuities. They promise a specified stream of equal cash payments each month to a lender. A 30-year mortgage with monthly payments is really a 360-payment annuity. (The "annu-ity" formula should really be called a "month-ity" formula in this case.) What would be your monthly payment if you took out a 30-year mortgage loan for $500,000 at a quoted interest rate of 7.5% per annum?

Before you can proceed further, you need to know one more bit of institutional knowledge here: Mortgage providers — like banks — quote interest by just dividing the mortgage quote by 12, so the true monthly interest rate is $7.5\%/12 = 0.625\%$. (They do not compound; if they did, the monthly interest rate would be $(1+7.5\%)^{1/12}-1 \approx 0.605\%$.)

A 30-year mortgage is an annuity with 360 equal payments with a discount rate of 0.625% per month. Its PV of $500,000 is the amount that you are borrowing. You want to determine the fixed monthly cash flow that gives the annuity this value:

(Omitted eq)

Solving for the cash flow tells you that the monthly payment on your \$500,000 mortgage will be \$500,000/143.018 ≈ \$3,496.07 for 360 months, beginning next month (time 1).

➤ Principal and Interest Components

There are two reasons why you may want to determine how much of your \$3,496.07 payment should be called interest payment and how much should be called principal repayment. The first reason is that you need to know how much principal you owe if you want to repay your loan early. The second reason is that Uncle Sam allows mortgage borrowers to deduct the interest, but not the principal, from their tax bills.

Here is how you can determine the split: In the first month, you pay 0.625% · \$500,000 = \$3,125 in mortgage interest. Therefore, the principal repayment is \$3,496.07 – \$3,125 = \$371.07 and the remaining principal is \$499,628.93. The following month, your interest payment is 0.625% · \$499,628.93 ≈ \$3,122.68 (note that your interest payment is now on the remaining principal), which leaves \$3,496.07 – \$3,122.68 = \$373.39 as your principal repayment, and \$499,255.54 as the remaining principal. And so on.

(Omitted solvenow)

Application: A Level-Coupon Bond

Let us exercise your newfound knowledge in a more elaborate example — this time with bonds. Recall that a bond is a financial claim sold by a firm or government. Bonds come in many varieties, but one useful classification is into coupon bonds and zero-bonds (short for zero coupon bond[zero coupon bonds]). A coupon bond pays its holder cash at many different points in time, whereas a zero-bond pays only a single lump sum at the maturity of the bond with no interim coupon. Many coupon bonds promise to pay a regular coupon similar to the interest rate prevailing at the time of the bond's original sale, and then return a "principal amount" plus a final coupon at the end of the bond.

For example, think of a coupon bond that will pay \$1,500 each half-year (semi-annual payment is very common) for 5 years, plus an additional \$100,000 in 5 years. This payment pattern is so common that it has specially named features: A bond with coupon payments that remain the same for the life of the bond is called a level-coupon bond. These are the most common bonds today. The \$100,000 here would be called the principal, in contrast to the \$1,500 semiannual coupon. Level bonds are commonly named by just adding up all the coupon payments over 1 year (here, \$3,000) and dividing this sum of annual coupon payments by the principal. Thus, this particular bond would be called a "3% semiannual coupon bond" (\$3,000 coupon per year divided by the principal of \$100,000). Now, the "3% coupon bond" is just a naming convention for the bond with these specific cash flow patterns — it is not the interest rate that you would expect if you bought this bond. In Section 2]sect:compounding-fvCompounding, we called such name designations interest *quotes*, as distinct from interest *rates*.

What should this \$100,000, 3% semiannual level-coupon bond sell for in late 2021? First, you should write down the payment structure for a 3% semiannual coupon bond. This comes from its defined promised payout pattern:

Year	Due Date	Bond Payment	Year	Due Date	Bond Payment
0.5	Jun 2022	$1,500	3.0	Dec 2024	$1,500
1.0	Dec 2022	$1,500	3.5	Jun 2025	$1,500
1.5	Jun 2023	$1,500	4.0	Dec 2025	$1,500
2.0	Dec 2023	$1,500	4.5	Jun 2026	$1,500
2.5	Jun 2024	$1,500	5.0	Dec 2026	$101,500

Second, you need to determine the appropriate rates of return that apply to these cash flows. In this example, assume that the prevailing interest rate is **5%** per annum. This translates into 2.47% for 6 months, 10.25% for 2 years, and so on.

Year	Maturity	Discount Rate	Year	Maturity	Discount Rate
0.5	6 Months	2.47%	3.0	36 Months	15.76%
1.0	12 Months	5.00%	3.5	42 Months	18.62%
1.5	18 Months	7.59%	4.0	48 Months	21.55%
2.0	24 Months	10.25%	4.5	54 Months	24.55%
2.5	30 Months	12.97%	5.0	60 Months	27.63%

Third, compute the discount factors, which are just $1/(1 + r_t) = 1/(1 + r)^t$, and multiply each future payment by its discount factor. This will give you the present value (PV) of each bond payment. From there, you can compute the bond's overall value:

Year	Due Date	Bond Payment	Rate of Return	Discount Factor	Present Value
0.5	Jun 2022	$1,500	2.47%	0.9759	$1,463.85
1.0	Dec 2022	$1,500	5.00%	0.9524	$1,428.57
1.5	Jun 2023	$1,500	7.59%	0.9294	$1,394.14
2.0	Dec 2023	$1,500	10.25%	0.9070	$1,360.54
2.5	Jun 2024	$1,500	12.97%	0.8852	$1,327.76
3.0	Dec 2024	$1,500	15.76%	0.8638	$1,295.76
3.5	Jun 2025	$1,500	18.62%	0.8430	$1,264.53
4.0	Dec 2025	$1,500	21.55%	0.8277	$1,234.05
4.5	Jun 2026	$1,500	24.55%	0.8029	$1,204.31
5.0	Dec 2026	$101,500	27.63%	0.7835	$79,527.91
				Sum:	$91,501.42

You now know that you would expect this 3% semiannual level-coupon bond to be trading for $91,501.42 today in a perfect market. Because the current price of the bond is below its named final principal payment of $100,000, this bond would be said to trade at a discount. (The opposite would be a bond trading at a premium. This happens when the bond pays more interest coupon than the prevailing interest-rate.)

The bond's value can be calculated more quickly via the annuity formula. Let's work in half-year periods. You have 10 coupon cash flows, each $1,500, at a per-period interest rate of 2.47%. According to the formula, these 10 coupon payments are worth

(Omitted eq)

In addition, you have the $100,000 repayment of principal, which will occur in year 5 and is therefore worth

(Omitted eq)

Together, the present values of the bond's cash flows again add up to $91,501.42.

Important Reminder of Quotes versus Returns: Never confuse a bond designation with the interest it pays. The "3% semiannual coupon bond" is just a designation for the bond's payout pattern. The bond will not give you coupon payments equal to 1.5% of your $91,501.42 investment (which would be $1,372.52). The prevailing interest rate (cost of capital) need not have anything to do with the quoted interest rate on the coupon bond. You could just as well determine the value of a 0% coupon bond, or a 10% coupon bond, given the prevailing 5% economy-wide interest rate. Having said all this, in the real world, many corporations choose coupon rates similar to the prevailing interest rate, so that at the moment of inception, the bond will be trading at neither a premium nor a discount. At least for this one brief at-issue instant, the coupon rate and the economy-wide interest rate may actually be fairly close. However, soon after issuance, market interest rates will move around, while the bond's payments will remain fixed, as designated by the bond's "3% semiannual coupon" name.

(Omitted solvenow)

3.3 The Formulas Summarized

(Omitted fig)

I am not a fan of memorization, but you must remember the growing perpetuity formula. You must also remember the annuity formula. They are used in many different contexts. There is also a growing annuity formula, which nobody remembers, but which you can look up if you ever need it:

(Omitted eq)

It is sometimes used in the context of pension cash flows, which tend to grow for a fixed number of time periods (T in the formula above) and then stop. However, even then it is not a necessary device. It is often more convenient and flexible to just work with the cash flows themselves within a spreadsheet.

Figure **??** summarizes the four special cash flows. The top graph shows the pattern of cash flows. For perpetuities, they go on forever. For annuities, they stop eventually. The bottom graph shows the present value of these cash flows. Naturally, these bars are shorter than those of their cash flows, which just means that there is a time value of money. The applicable formulas are below the graphs.

(Omitted solvenow)

Summary

This chapter covered the following major points:

- Figure **??** summarizes the four special cash flows and their quick valuation formulas.
- The PV of a simple perpetuity, which is a stream of constant cash flows that begin next period and that are to be discounted at the same annual cost of capital forever, is

  (Omitted eq)

- The PV of a growing perpetuity — with constant growth g, cash flows C beginning next year (time 1), and constant per-period interest rate r — is

  (Omitted eq)

- Stocks are often valued through an application of the growing perpetuity formula, called the Gordon dividend growth model.
- The PV of an annuity — T periods of constant C cash flows (beginning next year) and constant per-period interest rate r — is

  (Omitted eq)

- Fixed-rate mortgages are annuities. Interest rate quoted on such bonds are computed with the annuity formula.

Preview of the Chapter Appendix in the Companion

The appendix to this chapter in the companion (not here)

- shows how the annuity and perpetuity formulas can be derived.
- explains "equivalent annual costs" (which you already briefly encountered in Question). These allow you to compare projects with different rental periods — such as an 8-year lease that charges $1,000 per year and a 10-year lease that charges $900 per year.