Appendix Chapter. Technical Background

General Mathematical and Statistical Background

Exponents

- Finding a base:

$$3^2 = 9 \quad \Leftrightarrow 3 = 9^{1/2}$$
$$x^a = b \quad \Leftrightarrow \quad x = b^{1/a}$$

  A power of $1/2$ is also equivalent to the square root operation.

- Finding an exponent:

$$3^2 = 9 \Leftrightarrow 2 = \frac{\ln(9)}{\ln(3)}$$
$$a^x = b \Leftrightarrow x = \frac{\ln(b)}{\ln(a)}$$

  (Instead of the natural log ln, you could use any other log, too.)

☺Remember that kid who put his hand up in class and said "I will never use Algebra"? He is right! He will be the guy who will serve you your breakfast at the coffee shop counter. The thing that separates him from you is not the counter — it's algebra.

Summations

- Summation notation:

$$\sum_{i=1}^{N} f(i) = f(1) + f(2) + \cdots + f(N)$$

  This should be read as the "sum over all i from 1 to N." There are N terms in this sum. i is not a real variable: It is simply a dummy counter to abbreviate the notation. When 1 and N are omitted, it usually means "over all possible i."

- Summation rules:

$$\sum_{i=1}^{N}[a \cdot f(i) + b] = [a \cdot f(1) + b] + [a \cdot f(2) + b] + \cdots + [a \cdot f(N) + b]$$
$$= a \cdot \left[\sum_{i=1}^{N} f(i)\right] + N \cdot b$$

For example,

$$\sum_{i=1}^{3}\left[5 \cdot i^i + 2\right] = [5 \cdot 1^1 + 2] + [5 \cdot 2^2 + 2] + [5 \cdot 3^3 + 2]$$
$$= 7 + 22 + 137 = 166$$
$$= 5 \cdot \left(\sum_{i=1}^{3} i^i\right) + 3 \cdot 2 = 5 \cdot \left(1^1 + 2^2 + 3^3\right) + 6$$
$$= 5 \cdot 32 + 6 = 166$$

Linear Functions

- Linear functions: A function $\mathcal{L}(\cdot)$ is called a linear function if and only if $\mathcal{L}(a + b \cdot x) = a + \mathcal{L}(b \cdot x) = a + b \cdot \mathcal{L}(x)$, where a and b are constants.

  Here is an illustration. The (weighted) average is a linear function. For example, start with (5, 10, 15) as a data series. The average is 10. Pick $a = 2$ and $b = 3$. For averaging to be a linear function, it must be that

  $$\text{Average}(2 + 3 \cdot \text{Data}) = 2 + 3 \cdot \text{Average}(\text{Data})$$

  Let's try this — the left-hand side (LHS) would become the average of (17, 32, 47), which is 32. The right-hand side (RHS) would become $2 + 3 \cdot 10 = 32$. It works: Averaging indeed behaves like a linear function. In contrast, the square root is not a linear function, because $\sqrt{-2 + 3 \cdot 9} \neq -2 + 3 \cdot \sqrt{9}$. The LHS is 5, the RHS is 7. Linear functions are very important in financial economics:

  - Similar to averaging, expected values are linear functions. This is what has permitted us to interchange expectations and linear functions:

    $$\mathsf{E}(a + b \cdot X) = a + b \cdot \mathsf{E}(X)$$

    This will be expounded in the next section.
  - The rate of return on a portfolio is also a linear function of the investment weights. For example, a portfolio rate of return may be $r(x) = 20\% \cdot r_x + 80\% \cdot r_y$, where r_x is the rate of return on the component into which you invested \$20. For $r(x)$ to be a linear function, we need

    $$2 + 3 \cdot r(x) = r(2 + 3 \cdot x)$$
    $$a + b \cdot r(x) = r(a + b \cdot x)$$

    Substitute in

    $$2 + 3 \cdot (20\% \cdot r_x + 80\% \cdot r_y) = 20\% \cdot (2 + 3 \cdot r_x) + 80\% \cdot (2 + 3 \cdot r_y)$$

    Both sides simplify to $2 + 60\% \cdot r_x + 240\% \cdot r_y$, so our statement is true and a portfolio return is indeed a linear function.

However, not all functions are linear. The variance is not a linear function, because

$$\mathrm{Var}(a + b \cdot X) \neq a + b \cdot \mathrm{Var}(X)$$

You will confirm this in the next section.

Q A.1. If $(1 + x)^{10} = (1 + 50\%) = 1.5$, what is x?

Q A.2. If $(1 + 10\%)^{x} = (1 + 50\%) = 1.5$, what is x?

Q A.3. Are $\sum_{i=1}^{N} x_i$ and $\sum_{s=1}^{N} x_s$ the same?

Q A.4. In $\sum_{x=a}^{b} f(x,y)$, what are the variables?

Q A.5. Write out and compute $\sum_{x=1}^{3} (3 + 5 \cdot x)$. Is x a variable or just a placeholder to write the expression more conveniently?

Q A.6. Write out and compute $\left(\sum_{y=1}^{3} 3\right) + 5 \cdot \left(\sum_{y=1}^{3} y\right)$. Compare the result to the previous expression.

Q A.7. Is $\sum_{i=1}^{3} (i \cdot i)$ the same as $\left(\sum_{i=1}^{3} i\right) \cdot \left(\sum_{i=1}^{3} i\right)$?

Laws of Probability, Portfolios, and Expectations

Let's go over the algebra of probabilities and portfolios, which you had to use in the investments chapters. It is presented in a more mathematical fashion than it was in the chapters, which you may find easier or harder, depending on your background. If you have a statistics background, realize that our book's notation is simplified, because we do not place tildes over random variables.

☺Sadistics Background.

Single Random Variables

The **law of expectations** for single random variables are as follows:

- An expectation is defined as

$$\mathrm{E}(X) = \sum_{i=1}^{N} \mathrm{Prob}(i) \cdot [X = X(i)]$$

  It is basically a probability-weighted average.

☺Never ask Algebra about your X — she's gone!

- The expected value of a linear transformation (a and b are known constants):

$$\mathsf{E}(a \cdot X + b) = a \cdot \mathsf{E}(X) + b$$

To see this, consider a fair coin that can be either 1 or 2. Say $a = 4$ and $b = 10$. In this case, the LHS is $\mathsf{E}(a \cdot X + b) = \mathsf{E}(4 \cdot X + 10) = 0.5 \cdot (4 \cdot 1 + 10) + 0.5 \cdot (4 \cdot 2 + 10) = 0.5 \cdot 14 + 0.5 \cdot 18 = 16$. The RHS is $4 \cdot (0.5 \cdot 1 + 0.5 \cdot 2) + 10 = 16$. This all worked because expectation is a linear operator. (It is a fancy way of saying that it is a summation, which allows you to regroup the summation terms of the linear combination $a \cdot X + b$ inside the expectation, which is also a probability-weighted linear combination.) A little more generally, you could rename X as $f(X)$, so

$$\mathsf{E}(a \cdot f(x) + b) = a \cdot \mathsf{E}(f(x)) + b$$

However, you cannot always "pull" expectations in, so $\mathsf{E}(f(x))$ is not always $f(\mathsf{E}(X))$. For example, if $f(x) = x^2$, it is the case that

$$\mathsf{E}(X \cdot X) \neq \mathsf{E}(X) \cdot \mathsf{E}(X)$$

To see this, reconsider the fair "1 or 2" coin. The LHS is $\mathsf{E}(X^2) = 0.5 \cdot (1 \cdot 1) + 0.5 \cdot (2 \cdot 2) = 2.5$, but the RHS is $[\mathsf{E}(X)]^2 = (0.5 \cdot 1 + 0.5 \cdot 2)^2 = (1.5^2) = 2.25$.

- Definition of variance:

$$\mathsf{Var}(X) = \mathsf{E}(\left[X - \mathsf{E}(X)\right]^2)$$

It is sometimes easier to rewrite this formula as $\mathsf{Var}(X) = \mathsf{E}(X^2) - [\mathsf{E}(X)]^2$. Let me show you that this works. For our fair 1 or 2 coin example, the variance according to the main formula is $0.5 \cdot (1 - 1.5)^2 + 0.5 \cdot (2 - 1.5)^2 = 0.25$. For the second formula, we just computed $\mathsf{E}(X^2) = 2.5$ and $[\mathsf{E}(X)]^2 = 2.25$. Subtracting these terms yields the same 0.25.

- Definition of a standard deviation:

$$\mathsf{SD}(X) = \sqrt{\mathsf{Var}(X)}$$

- The variance of a linear combination (where a and b are known constants):

$$\mathsf{Var}(a \cdot X + b) = a^2 \cdot \mathsf{Var}(X)$$

For our fair 1 or 2 coin example, with $a = 4$ and $b = 10$, the LHS is $0.5 \cdot [(4 \cdot 1 + 10) - 16]^2 + 0.5 + 0.5 \cdot [(4 \cdot 2 + 10) - 16]^2 = 0.5 \cdot [-2]^2 + 0.5 \cdot [2]^2 = 4$. The RHS is $4^2 \cdot 0.25 = 4$.

Here is an extended illustration. A coin, whose outcome we call X, has 4 and 8 written on its two sides. These two outcomes can be written as $4 \cdot i$, where i is either 1 or 2. Therefore, the expected value of X is

$$\begin{aligned}
\mathsf{E}(X) &= \sum_{i=1}^{2} \mathsf{Prob}(X = (4 \cdot i)) \cdot (4 \cdot i) \\
&= \mathsf{Prob}(X = 4) \cdot (4) + \mathsf{Prob}(X = 8) \cdot (8) \\
&= 50\% \cdot 4 + 50\% \cdot 8 && = 6 \\
\mathsf{Var}(X) &= \sum_{i=1}^{2} \mathsf{Prob}(X = (4 \cdot i)) \cdot [(4 \cdot i) - 6]^2 \\
&= \mathsf{Prob}(X = 4) \cdot (4 - 6)^2 + \mathsf{Prob}(X = 8) \cdot (8 - 6)^2 \\
&= 50\% \cdot 4 + 50\% \cdot 4 && = 4
\end{aligned}$$

The standard deviation is the square root of the variance, here 2.

$\mathsf{E}(X^2)$ is, of course, not the same as $[\mathsf{E}(X)]^2 = [3]^2 = 9$, because

$$\begin{aligned}
\mathsf{E}(X^2) = \quad & \sum_{i=1}^{2} \mathsf{Prob}(X = (2 \cdot i)) \cdot (2 \cdot i)^2 \\
= \; & \mathsf{Prob}(X = 2) \cdot (2^2) + \mathsf{Prob}(X = 4) \cdot (4^2) \\
= \; & 50\% \cdot 4 + 50\% \cdot 16 = 10
\end{aligned}$$

Math with linear operators is much easier.

Now work with a linear transformation of the X, say, $Z = \$2.5 \cdot X + \$10$. This is a fundamental operation in finance, because the rates of return on portfolios are such linear transformations. For example, if you own 25% in A and 75% in B, you will earn $0.25 \cdot r_A + 0.75 \cdot r_B$. Thus,

Prob	**Coin**	X	Z
1/2	Heads	4	\$20
1/2	Tails	8	\$30

Math with linear operators is much easier.

You want to convince yourself that the expected value of Z, defined as $\$2.5{\cdot}X{+}\$10$, is $\$2.5 \cdot \mathsf{E}(X) + \$10 = \$25$. First, compute by hand the expected value the long way from Z,

$$\begin{aligned}
\mathsf{E}(Z) \;=\; & \sum_{i=1}^{2} \mathsf{Prob}(X = (4 \cdot i)\text{ , i.e., same as } Z = \$2.5 \cdot X + \$10) \cdot (Z_i) \\
=\; & \quad \mathsf{Prob}(X = 4\text{ , i.e., same as } Z = \$20) \cdot (\$20) \\
+\; & \quad \mathsf{Prob}(X = 8\text{ , i.e., same as } Z = \$30) \cdot (\$30) \\
=\; & 50\% \cdot \$20 + 50\% \cdot \$30 = \$25
\end{aligned}$$

Math with non-linear operators is harder.

Unlike the mean (the expected value), the variance is *not* a linear function. The variance of $Z = \$2.5{\cdot}X{+}\$10$ is *not* $\$2.5{\cdot}\mathsf{Var}(X) + \$10 = \$2.5{\cdot}4{+}\$10 = \$20$. Instead, $\mathsf{Var}(Z) = \mathsf{Var}(a \cdot X + b) = a^2 \cdot \mathsf{Var}(X) = (\$2.5)^2 \cdot \mathsf{Var}(X) = \$\$6.25 \cdot 4 = \$\$25$. You can confirm this working with Z directly:

$$\begin{aligned}
\mathsf{Var}(Z) \;=\; & \sum_{i=1}^{2} \mathsf{Prob}(X = (4 \cdot i)) \cdot [(Z_i) - \mathsf{E}(Z)]^2 \\
=\; & \mathsf{Prob}(X = 4\text{ , i.e., same as } Z = \$20) \cdot (\$20 - \$25)^2 \\
+\; & \mathsf{Prob}(X = 8\text{ , i.e., same as } Z = \$30) \cdot (\$30 - \$25)^2 \\
=\; & 50\% \cdot (\$5)^2 + 50\% \cdot (\$5)^2 = \$\$25
\end{aligned}$$

The standard deviation of Z is therefore $\sqrt{\$\$25} = \$5$.

Let us quickly confirm Formula A for $Z = \$2.5 \cdot X + \$10$:

$$\begin{aligned}
\$25 \;&=\; \mathsf{E}(\$2.5 \cdot X + \$10) = \$2.5 \cdot \mathsf{E}(X) + \$10 = \$2.5 \cdot 6 + \$10 = \$25 \\
\mathsf{E}(Z) = \;& \quad \mathsf{E}(a \cdot X + b) \quad = \quad a \cdot \mathsf{E}(X) + b
\end{aligned}$$

Let us also quickly confirm Formula A:

$$\begin{aligned}
\$\$25 \;&=\; \mathsf{Var}(\$2.5 \cdot X + \$10) = \$2.5^2 \cdot \mathsf{Var}(X) = \$\$6.25 \cdot 4 = \$\$25 \\
\mathsf{Var}(Z) = \;& \quad \mathsf{Var}(a \cdot X + b) \quad = \quad a^2 \cdot \mathsf{Var}(X)
\end{aligned}$$

Q A.8. What is the expected value and standard deviation of a bet B that pays off the number of points on a fair die, squared? For example, if the die lands on 3, you receive \$9.

Q A.9. Assume that you have to pay \$30, but you receive twice the outcome of the bet B from Question A.8. This is a new bet, called C. That is, your payoff is $C = -\$30 + 2 \cdot B$. What is the expected payoff and risk of your position? (Suggestion: Make your life easy by working with your answers from Question A.8.)

Portfolios

A portfolio is a set of weights in possible investment assets. That is, a portfolio is defined by its set of assets i, where each known investment is usually denoted as w_i. The rate of return r on a portfolio P is then

$$r_P = \sum_i w_i \cdot r_i$$

where r_i is the security return on security i. Portfolio returns are the weighted sum of multiple random variables.

- Portfolio return expectations:

$$\mathsf{E}\Big(\sum_i w_i \cdot r_i \Big) = \sum_i w_i \cdot \mathsf{E}(r_i)$$

  Although the weights are fixed and known constants, they cannot be pulled out of the summation, because they are indexed by i (each could be different from the others).

- Portfolio return riskiness:

$$\mathsf{Var}\Big(\sum_i w_i \cdot r_i \Big) = \sum_{i=1}^{N} \left\{ \sum_{j=1}^{N} \left[w_i \cdot w_j \cdot \mathsf{Cov}(r_i, r_j) \right] \right\} = \sum_{i=1}^{N} \sum_{j=1}^{N} \left[w_i \cdot w_j \cdot \mathsf{Cov}(r_i, r_j) \right]$$

  Of course, for the sake of better quantitative intuition, one would often compute the standard deviation by taking the square-root of the variance.

Coin Toss Example

Make sure you can work this yourself!!

Here is an illustration. Define a random variable, T, that is a coin toss outcome that will return either \$2 (heads) or \$4 (tails). Assume you have to pay \$2 to participate in one toss. This looks like a good bet: The rate of return on each coin toss will realize to be either zero or double. (You can't even lose money!) The *expected* rate of return on one toss is

$$\mathsf{E}(r_T) = E(P)/P_0 - 1 = (1/2 \cdot \$2 + 1/2 \cdot \$4)/\$2 - 1 = 50\%$$

If you could repeat this bet 100,000 times, you will have bet \$200,000 and ended up with approximately \$300,000, give or take.

The variance on *each* single coin toss is

$$\text{Var}(r_T) = 1/2 \cdot (0\% - 50\%)^2 + 1/2 \cdot (100\% - 50\%)^2 = 2{,}500\%\% = 0.25$$

Therefore, the standard deviation of each coin toss is $\sqrt{2{,}500\%\%} = \sqrt{0.25} = 50\%$.

Multiple independent bets.

Now, bet on a "portfolio" of two independent bundled coin toss outcomes. Say you have to invest \$10 on the first coin's toss ($w_1 = \$10$) and \$20 on the second coin's toss ($w_2 = \$20$). Your portfolio is $\{w_1, w_2\} = \{\$10, \$20\}$. You can also compute your portfolio's investment weights instead of its absolute investments:

$$w_1 = \frac{\$10}{\$30} \approx 0.33 \quad \text{and} \quad w_2 = (1 - w_1) = \frac{\$20}{\$30} \approx 0.67$$

Your overall "two-toss portfolio" rate of return is

$$r = \sum_{i=1}^{2} w_i \cdot r_i$$

The portfolio expected return.

We can now use the formulas to compute your risk and reward. To compute your expected portfolio rate of return, use

$$\begin{aligned} \text{E}(r) = \sum_{i=1}^{2} w_i \cdot \text{E}(r_i) &= w_1 \cdot \text{E}(r_1) + w_2 \cdot \text{E}(r_2) \\ &= 1/3 \cdot (50\%) + 2/3 \cdot (50\%) = 50\% \end{aligned}$$

(Recall that an expectation is a linear operator, that is, a summation. A portfolio is a summation, too. Because both are ultimately nothing but summations, you can regroup terms, which means that the above formula works.)

Diversification of independent bet outcomes.

To compute your variance, use

$$\begin{aligned} \text{Var}(r) &= \sum_{i=1}^{2}\sum_{j=1}^{2} w_i \cdot w_j \cdot \text{Cov}(r_i, r_j) \\ &= w_1 \cdot w_1 \cdot \text{Cov}(r_1,r_1) + w_1 \cdot w_2 \cdot \text{Cov}(r_1,r_2) \\ &\quad + w_2 \cdot w_1 \cdot \text{Cov}(r_2,r_1) + w_2 \cdot w_2 \cdot \text{Cov}(r_2,r_2) \\ &= w_1^2 \cdot \text{Cov}(r_1,r_1) + 2 \cdot w_1 \cdot w_2 \cdot \text{Cov}(r_1,r_2) + w_2^2 \cdot \text{Cov}(r_2,r_2) \\ &= w_1^2 \cdot \text{Var}(r_1) + 2 \cdot w_1 \cdot w_2 \cdot \text{Cov}(r_1,r_2) + w_2^2 \cdot \text{Var}(r_2) \\ &= (1/3)^2 \cdot \text{Var}(r_1) + 2 \cdot w_1 \cdot w_2 \cdot 0 + (2/3)^2 \cdot \text{Var}(r_2) \\ &= (1/9) \cdot \text{Var}(r_1) + (4/9) \cdot \text{Var}(r_2) \\ &= (1/9) \cdot 0.25 + (4/9) \cdot 0.25 \\ &\approx 0.1389 \end{aligned}$$

The standard deviation is therefore $\sqrt{0.1389} \approx 37.3\%$. This is lower than the 50% risk that a "single coin toss" portfolio would have.

Q A.10. Repeat the example, but assume that you invest \$15 into each coin toss rather than \$10 and \$20, respectively. Would you expect the risk to be higher or lower? (Hint: What happens if you choose a portfolio that invests more and more into just one of the two bets?)

Cumulative Normal Distribution Table

z	$\mathcal{N}(z)$	z	$\mathcal{N}(z)$	z	$\mathcal{N}(z)$	z	$\mathcal{N}(z)$	z	$\mathcal{N}(z)$	z	$\mathcal{N}(z)$
–4.0	0.00003										
–3.5	0.00023										
–3.0	0.0013	–2.0	0.0228	–1.0	0.1587	0.0	0.5000	1.0	0.8413	2.0	0.9772
–2.9	0.0019	–1.9	0.0287	–0.9	0.1841	0.1	0.5398	1.1	0.8643	2.1	0.9821
–2.8	0.0026	–1.8	0.0359	–0.8	0.2119	0.2	0.5793	1.2	0.8849	2.2	0.9861
–2.7	0.0035	–1.7	0.0446	–0.7	0.2420	0.3	0.6179	1.3	0.9032	2.3	0.9893
–2.6	0.0047	–1.6	0.0548	–0.6	0.2743	0.4	0.6554	1.4	0.9192	2.4	0.9918
–2.5	0.0062	–1.5	0.0668	–0.5	0.3085	0.5	0.6915	1.5	0.9332	2.5	0.9938
–2.4	0.0082	–1.4	0.0808	–0.4	0.3446	0.6	0.7257	1.6	0.9452	2.6	0.9953
–2.3	0.0107	–1.3	0.0968	–0.3	0.3821	0.7	0.7580	1.7	0.9554	2.7	0.9965
–2.2	0.0139	–1.2	0.1151	–0.2	0.4207	0.8	0.7881	1.8	0.9641	2.8	0.9974
–2.1	0.0179	–1.1	0.1357	–0.1	0.4602	0.9	0.8159	1.9	0.9713	2.9	0.9981
										3.5	0.99977
										4.0	0.99997

Table A.1: ***Cumulative Normal Distribution Table.*** Normal score (z) versus standardized normal cumulative distribution probability $\mathcal{N}(z)$

Table A.1 allows you to determine the probability that an outcome X will be less than a prespecified value x, when standardized into the score z, if X (and thus z follow a **normal distribution**). For example, if the mean is 15 and the standard deviation is 5, an outcome of X = 10 is 1 standard deviation below the mean. This standardized score can be obtained by computing $z(x) = [x - \mathsf{E}(x)]/\mathsf{SD}(x) = (x - 15)/5 = (10 - 15)/5 = (-1)$. This table then indicates that the probability that the outcome of X (i.e., drawn from this distribution with mean 15 and standard deviation 5) will be less than 10 (i.e., less than its score of z = –1) is 15.87%.

Figure A.2 shows what the table represents. Figure A.2(a) shows the classical bell curve. Recall that at z = –1, the table gives $\mathcal{N}(z = -1) = 15.87\%$. This 15.87% is the shaded area under the curve up to and including z = –1. Figure A.2(b) just plots the values in the table itself, that is, the area under the graph to the left of each value from Figure A.2(a).

S-curve is often more useful than the bell curve.

If you ever need to approximate the cumulative normal distribution in a spreadsheet (like Google Sheets, Excel, or Libreoffice), you can use the built-in function **=normsdist()**.

Compiled: Friday 9th December, 2022

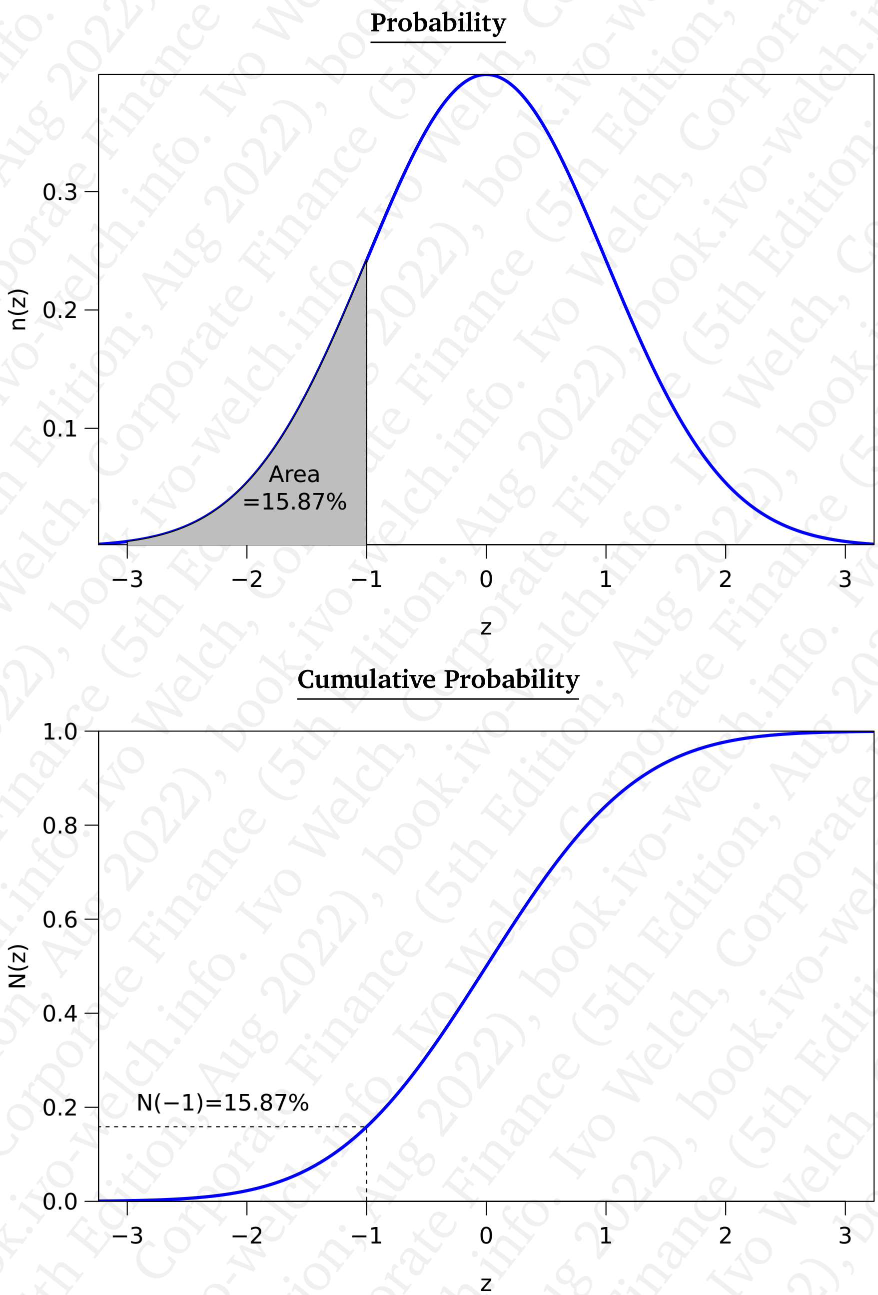

Figure A.2: ***The Standard Normal Distribution.***