Perpetuities

(Welch, Chapter 03-A)

Ivo Welch

Mon Jun 20 23:34:39 2022

Maintained Assumptions

- We assume **perfect markets**:
  1. No differences in opinion.
  2. No taxes.
  3. No transaction costs.
  4. No big sellers/buyers—infinitely many clones that can buy or sell.
- We again assume **perfect certainty**, so we know what the RoR is on every project.
- We assume constant RoRs (per year).

General Questions

- Are there any shortcut NPV formulas for long-term projects—at least under certain common assumptions?
- Or, do we always have to compute long summations for projects with many, many periods?
- Why do some of the folks have the magic ability to quickly tell you estimates that would take you hours to figure out with the NPV formula?

Specific Sample Questions

- What is the value of a firm that generates $1 million in earnings per year and grows by the inflation rate?
- If your firm earns $5 million/year, and the interest rate is 5%, what is its approximate value?
- What is a Pro-Forma terminal market-value estimate?

Simple Perpetuities

A **Perpetuity** is a financial instrument that pays C dollars per period *forever*.

- If the interest rate is constant and the first payment from the perpetuity arrives in period 1,

$$PV(C,r) = \sum_{t=1}^{\infty} \frac{C}{(1+r)^t} = \frac{C}{r} .$$

- This notation is *very common* in finance:
  - C and r are the two real input variables.
  - t is an ephemeral counter (not an input variable).

Perpetuity Footnotes

Make sure you know when the first cash flow begins: Tomorrow [t=1], not today [t=0]!

- I sometimes write C_1/r to remind myself of timing, even though cash flows are the same at time 1 as they are at time 25—I could have written C_{25} instead.

(NFL) Booth Review

Write out the formula $\sum_{t=1}^{\infty} \frac{C}{(1+r)^t}$:

Programming Language

$\sum_{t=1}^{T} f(t)$ is

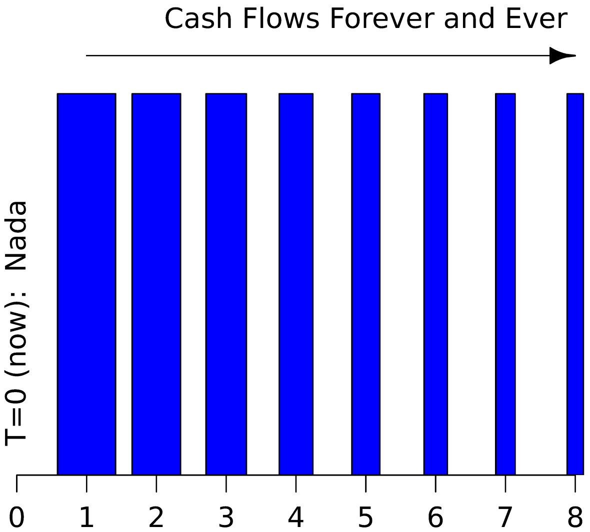
```
function sum(integer T)
   sumup <- 0.0
   for t from 1 to T
      sumup <- sumup + f(T)
   end
   return sumup
end
```

Infinite Sums?

How can an infinite sum be worth less than infinite cash?

- Because each future C is worth *a lot* less than the preceding C.
- In the graph on the next page, the PV of each cash flow is the bar's *area*.
- Soon, terms *add almost* nothing.

Graph: Perpetuity



Value of Perpetuity I

What is the value of an unbreakable promise to receive $10 forever, beginning next year, if the interest rate is 5% per year?

Value of Perpetuity II

What is the value of an unbreakable promise to receive $10 forever, beginning **this** year, if the interest rate is 5% per year?

Perpetuity Formula Mod

What is the perpetuity formula if the first cash flow starts today rather than tomorrow?

Nerd: Time Consistency

- Assume an interest rate of 10%.
- A perpetuity today with \$1 forever is worth \$ \$1/0.1= \$10 \$.
- A perpetuity tomorrow with \$1 forever will be worth \$1/0.1= \$10 tomorrow.
- Today's perpetuity gives you \$1 extra next period, *and* leaves you with *a* then \$10 perpetuity. At 10%, they *are* worth \$1/(1+10%) *and* and \$10/(1+10%), respectively. The latter is next year's perpetuity.

Growing Perpetuities

A growing perpetuity pays

- C next year
- then $C \cdot (1 + g)$ the following year,
- then $C \cdot (1 + g)^2$ the following year,
- then ...

Growing perpetuities generalize simple perpetuities ($g = 0$).

Growing Perpetuity Table of Cash Flow and Present Values, g=10%

Time	Cash Flow	Is Worth Today
0	$0	$0
1	$100	$100
2	$100*1.1	$110
3	$100*1.1^2	$121
4	$100*1.1^3	$133
...	...	...
t	$100*1.1^t	...
...	...	...

Growing Perpetuities Formula

The PV of a growing perpetuity is

$$PV(C_1, g, r) = \sum_{t=1}^{\infty} \frac{C_1 \cdot (1+g)^{t-1}}{(1+r)^t} .$$

The real beauty is the shortcut formula,

$$PV(C_1, g, r) = \frac{C_1}{r-g} .$$

Growing Perpetuities Footnotes

You must memorize the shortcut formula, and know what it means!

- The growth term g acts like a reduction in the interest rate r.
- The time subscript for the payment matters now, because $C_1 \neq C_2 \neq C_t$.

(NFL) Booth Review

Check the growing perpetuity formula by hand!

Infinite Sums?

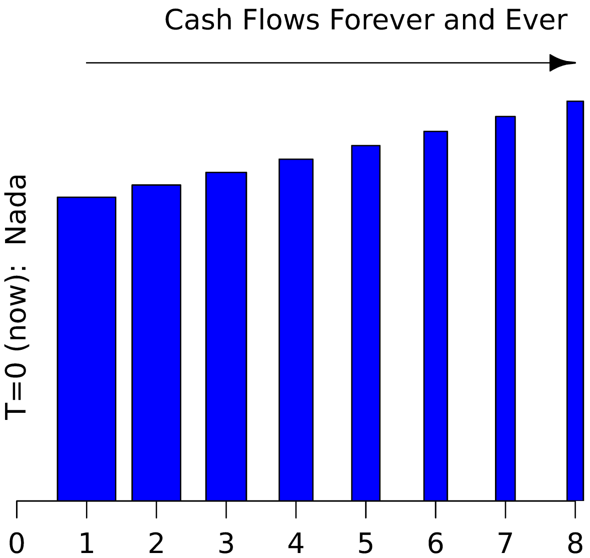
How can an infinite sum be worth less than infinite cash?

- Because the growth g is not too fast.
- Each rectangle is smaller than the preceding one, i.e., each PV is smaller than the preceding one.

What if $g \geq r$?

- The formula then makes no sense.

Graph: Growing Perpetuity



Value of Eternal Guarantee

What is the value of a guarantee to receive \$10 next year, growing by 2%/year (just the inflation rate) forever, if the interest rate is 6%/year?

Growing Formula Mod

What is the value of *a* firm that just paid $10 **this** year, growing by 2%/year forever, if the interest rate is 5%/year?

Example PV Calc I

What is the formula for the value of a firm which will only grow at the inflation rate, and which will have $1 million of earnings next year?

Example PV Calc II

- In 10 years, a firm will have annual cash flows of $100 million.
- Thereafter, its cash flows will grow at the inflation rate of 3%.
- If the applicable interest rate is 8%, estimate its value if you will sell the firm in 10 years?
- What would this "terminal value" be worth today?

Pro Forma TVE

Terminal Value Estimates are the most common use of the formula:

- guesstimate the PV of the firm after an arbitrary T years in the future.
- The inflation rate is often the common long-run growth rate, g.
- A typical T in a "pro-forma" would be 5-10 years.

Gordon Dividend Growth Model

What should be the share price of a firm that

- pays dividends of \$1/year,
- whose dividends grow by 4% every year, and
- which will continue to do so forever,
- if its cost of capital (CoC) is 12%/year?

GDGM for CFAs

CFA Exam: Using D for C gives you the GDGM (Gordon Dividend Growth Model):

$$P = \frac{D}{r - g} .$$

Ergo $D/P = r - g$.

GDGM for Real

Don't trust the GDGM

- Firms can shift dividends!
- What a firm does not pay out in dividends today will make more hey (dividends) tomorrow.
- It should not matter if the firm cancels its $1 dividends this year in order to pay out an extra $1.05 next year.

GDGM Improvements?

- An improvement uses the **plowback ratio**:
  - it takes into account that reinvested cash should pay more dividends in the future,
  - but it's still just lipstick on a pig.
- A better valuation formula could use earnings instead of dividends,
  - because earnings are more difficult to shift around.

GDGM Implied Cost of Capital (ICC)

What is the CoC for a firm that

- pays a dividend yield (D/P) of 5%/year today,
- if its dividends are expected to grow at a rate of 3%/year forever?

GDGM ICC Formula

An *Implied Cost of Capital* (*ICC*) is the expected RoR embedded in the stock price today.

- GDGM is sometimes used to estimate an implied cost of capital, ICC,
- via the inverted formula $r = D/P + g$.
- A higher P today implies a lower implied CoC at which the firms can obtain capital from investors.

Advanced Finance

- Possible to show that a formula like the GDGM is more general
- If a firm has a high D/P, it must mean that it will either have a high r or a low g.
  - If a firm has a div yield of 50%, if this div yield will not go down, we will make a huge r.
  - If a firm has a div yield of 0%, if this div yield will not go up, we will make a lousy r.

S&P500 ICC

- The current P/E ratio of stocks is, e.g., at https://www.currentmarketvaluation.com/models/price-earnings.php.
  - In 2022, this was about 30 for the S&P500.
  - The historical average was about 20. (30 is about one sd more.)
- The growth rate of earnings is, e.g., at https://www.multpl.com/s-p-500-earnings-growth/table/by-year.

- If future GDP growth rate is about 3-5% (the growth rate of GDP), what should investors reasonably expect about future RoR implied by a P/E ratio of 30?

Quick Calc: Value of Firm

Our firm has earned $100,000 this year.

It has stopped growing in *real* terms.

The current interest rate is 6%/year.

The inflation rate is 2%/year.

What is the value of our firm?

- What is it over-the-envelope ?
- What is it exactly?
- What is the first cash flow?

Growth Rate of Google

- In May 2022, Alphabet (Google)'s share price was about \$2,200. Its EPS was about \$110.
- Therefore, Google's P/E Ratio was about 20.
- Google's CoC for equity was about 6%/y.
- What does the market believe G's *as-if-eternal* earnings growth rate will be?

Metaphysics

Are perpetuities meaningful?

- How long will firms last?
- How long will Google last?
- What firms or institutions have survived from the Roman empire?